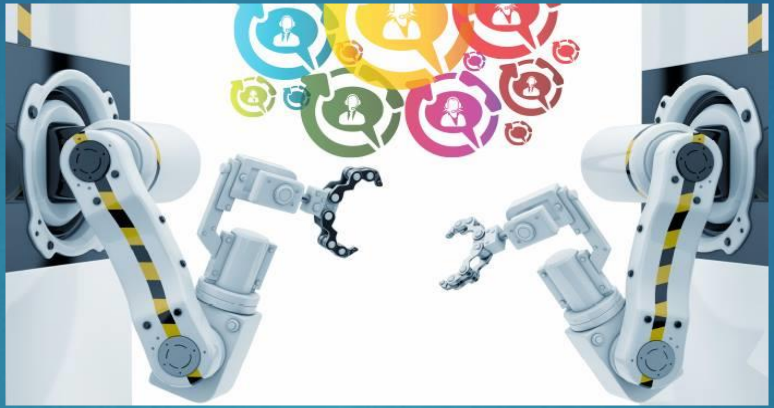


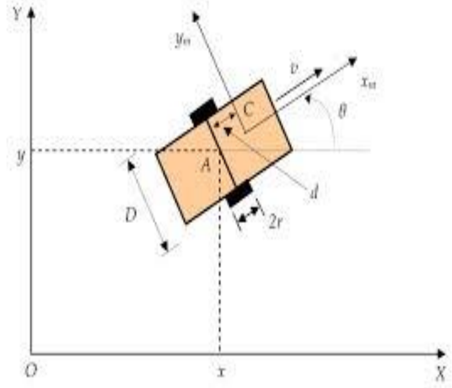
Robot Forward Kinematics

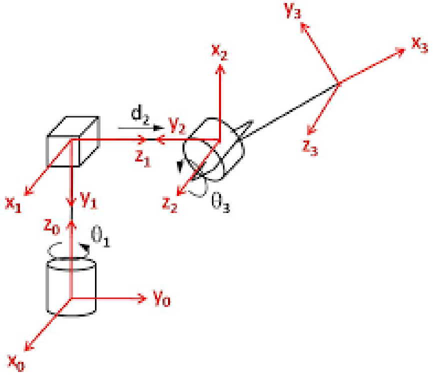
MUHAMMAD A. SHEHAB

Introduction

- Robot kinematics:
  - Description of motion of the robot without consideration of the forces and torques causing the motion.
  - The Kinematics is a geometric description.

Kinematics
- Forward Kinematics
- Inverse Kinematics





Forward and Inverse Kinematics

- Forward Kinematics
  - Determination of the (actual) position and orientation of the end-effector given the values for the joint variables of the robot
- Inverse Kinematics
  - Determination of the values of the joint variables of the robot given the (desired) position and orientation of the end-effector
- Notes
  - Inverse Kinematics is required to determine the control action
  - Forward kinematics is required to give feedback about end-effector pose

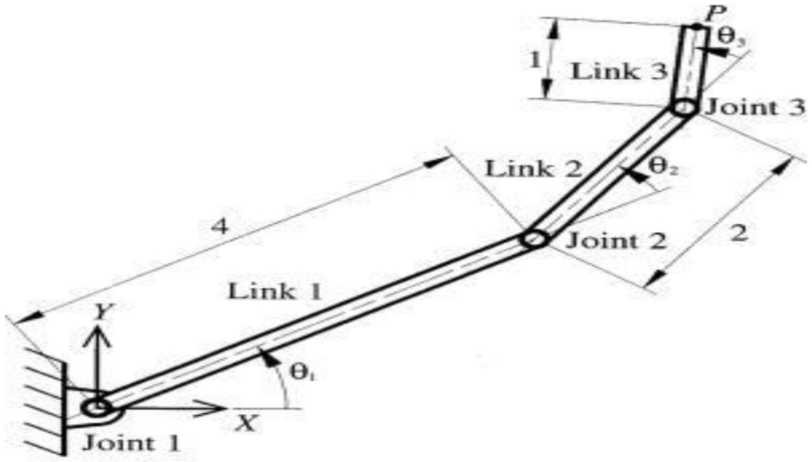
Kinematic Chains

MANIPULATORS

FORWARD KINEMATICS

Robot Manipulators

- A robot manipulator is composed of a set of links connected together by joints.
- A robot manipulator with n joints will have $n + 1$ links as each joint connects two links.

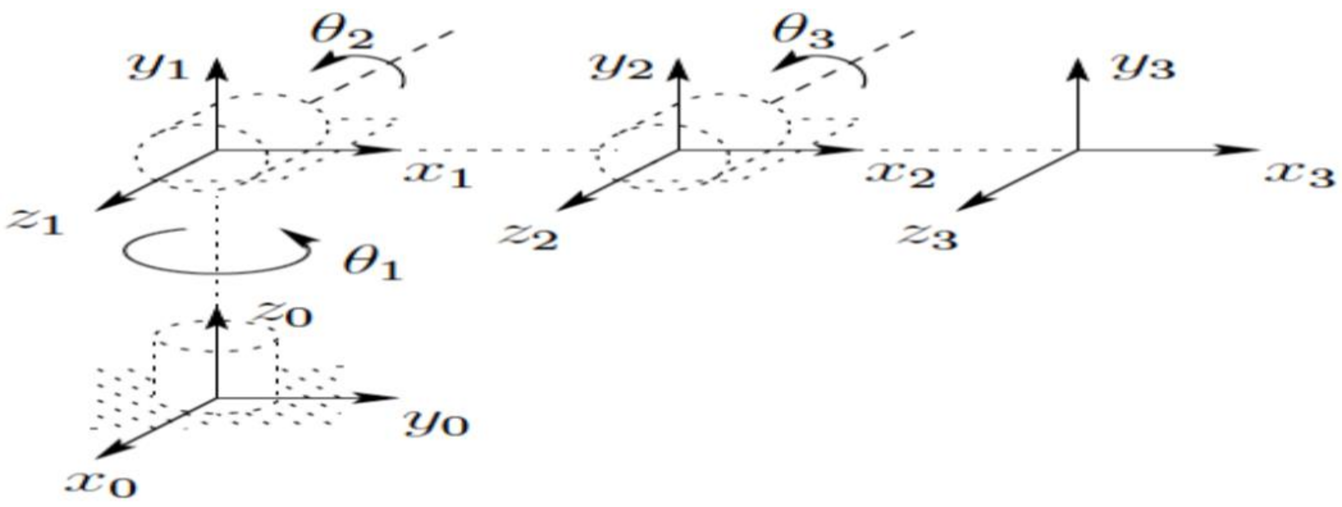


Robot Manipulators

- Convention
  - We number the joints from 1 to n , and we number the links from 0 to n , starting from the base.
  - By this convention, joint i connects link $i-1$ to link i. We will consider the location of joint i to be fixed with respect to link $i-1$.
  - When joint i is actuated, link i moves.
  - With i^{th} joint we associate a joint variable, denoted by q_i
    - Angle of rotation in case of revolute joint
    - Joint displacement in case of prismatic joint

Kinematics Analysis

- To perform the kinematic analysis, we attach a coordinate frame rigidly to each link. In particular, we attach frame $o_i x_i y_i z_i$ to link i.
- The frame $o_0 x_0 y_0 z_0$, which is attached to the robot base, is referred to as the inertial frame.



Kinematic Analysis

- Suppose A_i is the homogeneous transformation matrix that expresses the position and orientation of $o_i x_i y_i z_i$ with respect to $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$.
- A_i is a function of only a single joint variable, namely q_i.
  - $A_i = A_i(q_i)$
- The homogeneous transformation matrix that expresses the position and orientation of frame $o_j x_j y_j z_j$ with respect to frame $o_i x_i y_i z_i$ is denoted by:

$$T_j^i = \begin{cases} A_{i+1} A_{i+2} \ldots A_{j-1} A_j & \text{if } i < j \\ I & \text{if } i = j \\ (T_i^j)^{-1} & \text{if } j < i \end{cases}$$

End-Effector Pose

- The position and orientation of the end-effector with respect to the inertial frame are denoted by a vector O_n^0 (represents the coordinates of the origin of the end-effector frame with respect to the base frame)and a rotation matrix R_n^0 respectively.
- The homogeneous transformation matrix of end-effector pose H is:

$$H = \begin{bmatrix} R_n^0 & O_n^0 \\ 0 & 1 \end{bmatrix}$$

$$H = T_n^0 = A_1(q_1) \ldots \ldots A_n(q_n), \qquad A_i = \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0 & 1 \end{bmatrix}$$

Pose components

- For homogeneous transformation matrix T_j^i

$$T_j^i = A_{i+1}A_{i+2} \dots \dots A_j = \begin{bmatrix} R_j^i & O_j^i \\ 0 & 1 \end{bmatrix}, \qquad j > i$$

$$R_j^i = R_{i+1}^i R_{i+2}^{i+1} \dots \dots R_j^{j-1}$$

$$O_j^i = O_{j-1}^i + R_{j-1}^i O_j^{j-1}$$

Review Example

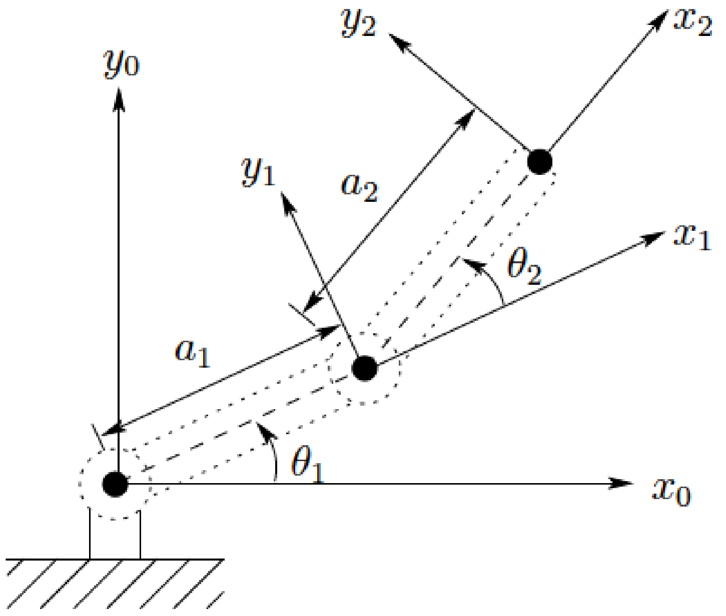
- Consider the two-link planer manipulator

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^0 = A_1$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Note: $c_1 \equiv \cos(\theta_1)\,, c_{12} \equiv \cos(\theta_1 + \theta_2)$

$s_1 \equiv \sin(\theta_1)\,, s_{12} \equiv \sin(\theta_1 + \theta_2)$

DH-Convention

DH ASSUMPTIONS

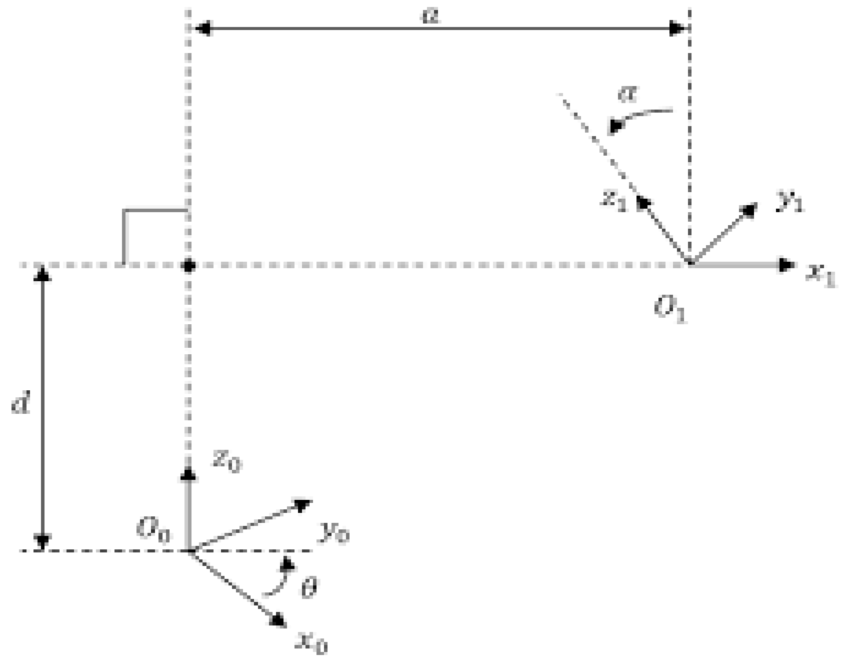
ASSIGNING FRAMES

Denavit-Hartenberg convention

- Denavit-Hartenberg convention or DH convention is a commonly used method for selecting reference frames of robots
- In DH convention, the 6 parameters associated with an arbitrary homogeneous transformation are reduced to 4 by appropriate selection of reference frames
- In this convention, each homogeneous transformation A_i is represented as a product of four basic transformations using the following parameters:
  - a_i: Link length
  - α_i: Link twist
  - d_i: Link offset
  - θ_i: joint angle

DH-Assumptions

- DH coordinate frame assumptions
  - DH1- The axis x_i is perpendicular to the axis z_{i-1}
  - DH2- The axis x_i intersects the axis z_{i-1}
- Under the above assumptions A_i is achieved by
  1. $Rot(z, \theta_i)$
  2. $Trans(z, d_i)$
  3. $Trans(x, a_i)$
  4. $Rot(x, \alpha_i)$
- $A_i = Rot(z, \theta_i) * Trans(z, d_i) * Trans(x, a_i) * Rot(x, \alpha_i)$



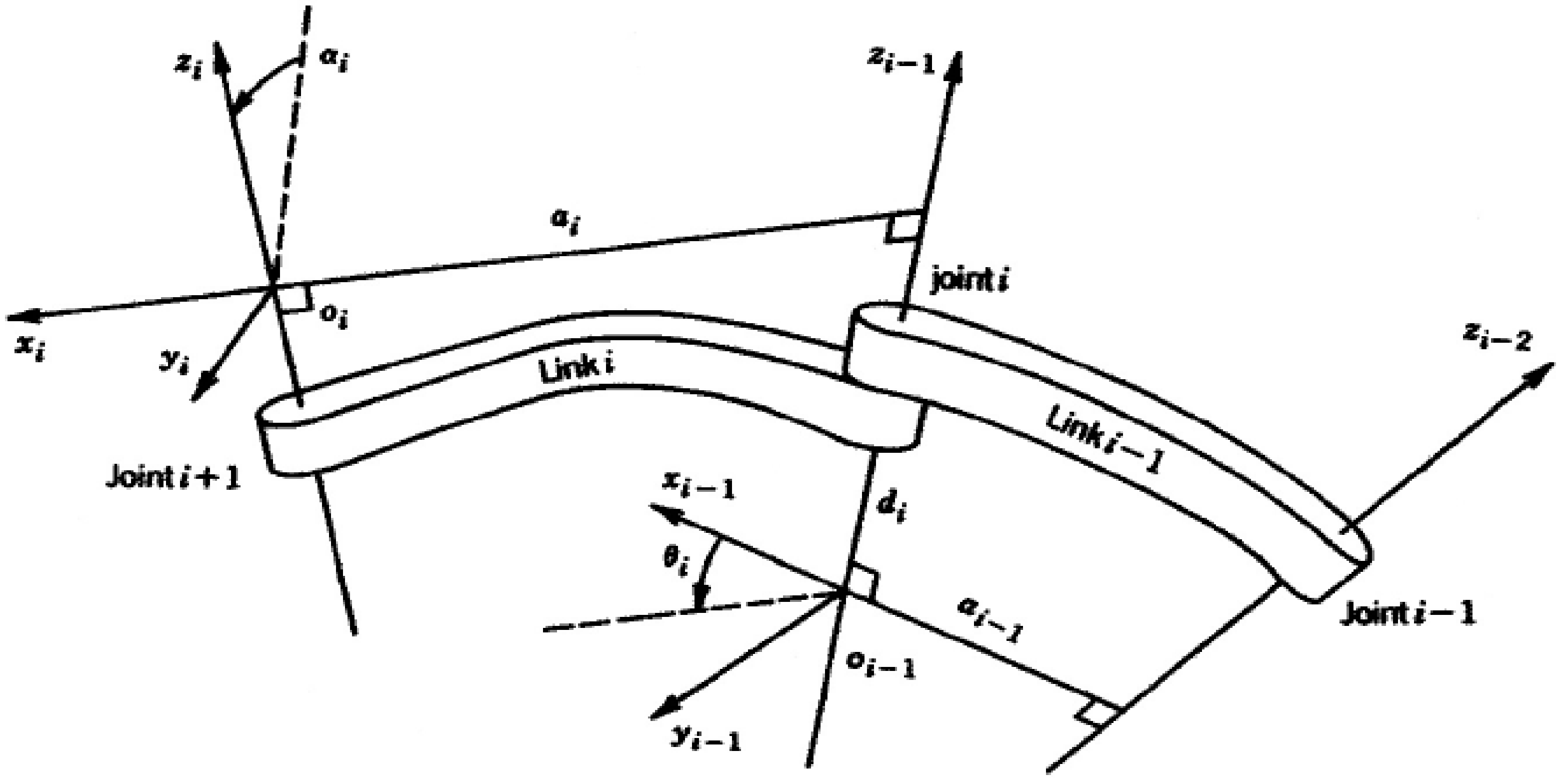
DH-Assumptions

$$
\begin{aligned}
A_i &= Rot_{z,\theta_i}\,\mathrm{Trans}_{z,d_i}\,\mathrm{Trans}_{x,a_i}\,Rot_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

DH-Parameters

- The parameter a is the distance between the axes z_0 and z_1 measured along the axis x_1
- The parameter α is the angle between the axes z_0 and z_1, measured in a plane normal to x_1 axis
- The parameter d is the perpendicular distance from the origin O_0 to the intersection of the x_1 axis with z_0 axis measured along the z_0 axis
- The parameter θ is the angle between x_0 axis and x_1 axis measured in a plane normal to z_0 axis
- Hint:
  - d: is, only, the variable in case of prismatic joints
  - θ: is, only, the variable in case of revolute joints

Coordinate Frames Assignment



Assignment steps

1. Assign z_i to be axis of actuation of joint $i + 1$
2. Establish arbitrarily the base frame: $x_0, y_0,$ (z_0 determined in step 1)
3. Define x_i based on one of three cases
   a. The axes z_{i-1} and z_i intersects
   b. The axes z_{i-1} and z_i are parallel
   c. The axes z_{i-1} and z_i are not coplanar
4. Define y_i in the appropriate direction to complete the frame
5. The final coordinate frame $o_n x_n y_n z_n$ is commonly referred to as the end-effector or tool frame. The origin O_n is most often placed symmetrically between the fingers of the gripper

Assignment of x_i axis

z_{i-1} and z_i are intersected

- x_i is chosen normal to the plane formed by z_{i-1} and z_i .
- The positive direction of x_i is arbitrary.
- The most natural choice of o_i to be at the intersection point of z_{i-1} and z_i
- In this case a_i equal zero

z_{i-1} and z_i are parallel

- There are infinitely common normal between them
- DH1 doesn't specify x_i completely
- It's free to choose o_i anywhere along z_i
- The normal going through o_i is chosen to be x_i

z_{i-1} and z_i not coplanar

- There exists a unique shortest line segment from z_{i-1} to z_i, perpendicular to both of them
- This line segment defines x_i
- The point where the line of x_i intersects z_i is the origin o_i

Examples

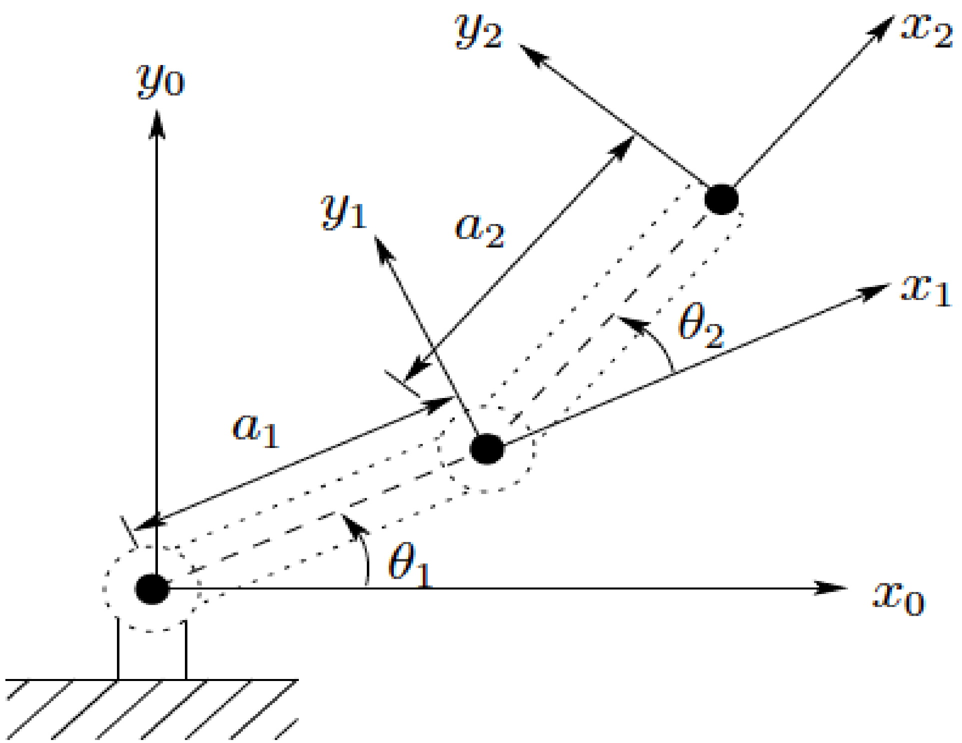
PLANAR MANIPULATOR

CYLINDRICAL ROBOT

DH-solution steps

- Assignment of coordinate frames according to the defined rules
- Construction of DH-Table containing the parameters of each transformation matrix between two successive links
- Compute the transformation matrix A_i for each link
- Compute the whole homogenous transformation matrix T_n^0
- Remember:
  - $A_i = Rot(z, \theta_i) * Trans(z, d_i) * Trans(x, a_i) * Rot(x, \alpha_i)$
  - $H = T_n^0 = A_1(q_1) \dots \dots A_n(q_n)$

Two-Link Planar Manipulator



Link parameters for 2-link planar manipulator

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

Two-Link Planar Manipulator

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

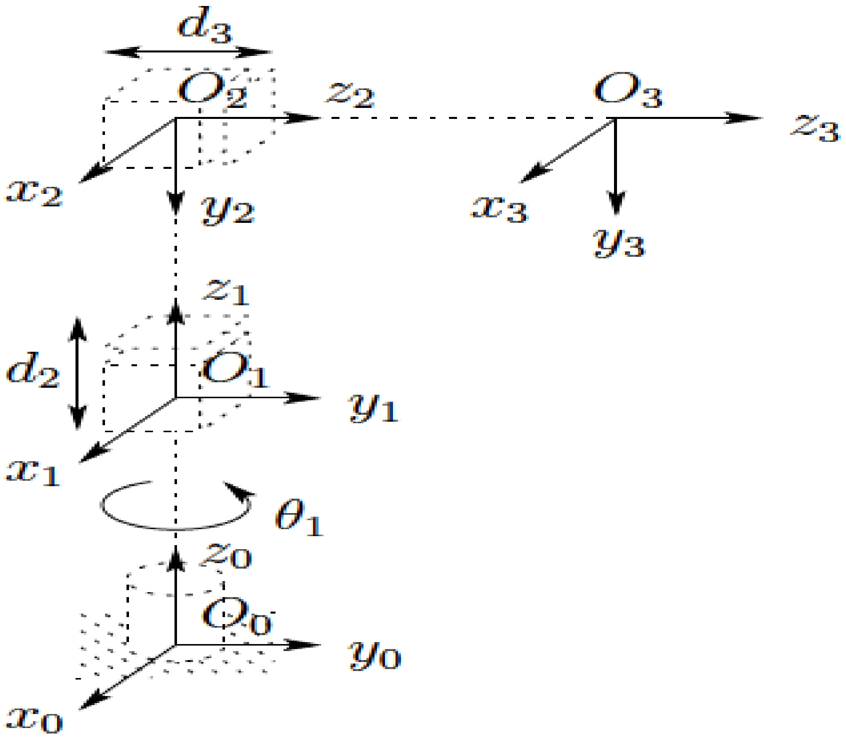
$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The T-matrices are thus given by

$$T_1^0 = A_1$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Three-Link Cylindrical Robot



Link parameters for 3-link cylindrical manipulator

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

Three-Link Cylindrical Robot

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reference

- Mark W. Spong, Seth Hutchinson and M. Vidyasagar, "Robot Modelling and Control", Wiley, 2005